EECS 20N: Structure and Interpretation of Signals and Systems Problem Set 2
Department of EECS Issued: 6 February 2007
UNIVERSITY OF CALIFORNIA BERKELEY **Due: 13 February 2007, 5pm**

> Suppose you want to teach the "cat" concept to a very young child. Do you explain that a cat is a relatively small, primarily carnivorous mammal with retractile claws, a distinctive sonic output, etc.?
>
> I'll be not. You probably show the kid a lot of different cats, saying "kitty" each time, until it gets the idea.
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> To put it more generally, generalizations are best made by abstractions from experience. They should come one at a time; too many at once overload the circuits.[a]
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> [a] Excerpts from *Can We Make Mathematics Intelligible?*, by Ralph P. Boas, The American Mathematical Monthly, Vol. 88, No. 10, December 1981, pp. 727–731.

Policy Statement

- You are encouraged to collaborate, but only in a group of up to *five* current EECS 20N students.
- On the solution document that you turn in for grading, you must write the names of your collaborators below your own; each teammate must submit for our evaluation a distinct, self-prepared solution document containing original contributions to the collaborative effort.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- Unless we explicitly state otherwise, you will receive full credit *only if* you explain your work succinctly, but clearly and convincingly.
- Typically, we evaluate your solutions for only a subset of the assigned problems. A priori, you do not know which subset we will grade. It is to your advantage to make a bona fide effort at tackling *every* assigned problem.
- If you are asked to "sketch" a function, it means that you must provide a hand-drawn sketch, and not a plot generated by a computing device.

Overview

Among other things, this problem set is designed to improve your fluency with the mathematical foundations of this course, for example, logical models and deductive reasoning; and set-theoretic and functional models for, and representations and properties of, signals and systems.

Subject Matter

Appendix A and Chapters 1 and 2 of the textbook (Lee & Varaiya). Lectures up to, and including, the Lecture 8 on 8 February 2007.

HW2.1 For each function described below, explain whether the function is one-to-one (injective), onto (surjective), both (bijective), or neither. If the function is bijective (onto *and* one-to-one), determine the inverse of the function.

(a) $f : \mathbb{R} \to \mathbb{R}\,,\ \ \forall x \in \mathbb{R}\,, f(x) = e^{-|x|}$.

(b) $\mathsf{Sinc} : \mathbb{R} \to \mathbb{R}\,,\ \ \forall t \in \mathbb{R}\,, \mathsf{Sinc}(t) = \dfrac{\sin(\pi\, t)}{\pi\, t}$.

(c) $h : \mathsf{A} \to \mathsf{B}$, where $\mathsf{A} = \{Q, R, S, T\}$, $\mathsf{B} = \{1, 2, 3\}$, and $\mathsf{graph}(h) = \{(Q, 1), (R, 1), (S, 2), (T, 3)\}$.

(d) $r : \mathsf{A} \to \mathsf{B}$, where $\mathsf{A} = \{Q, R, S\}$, $\mathsf{B} = \{1, 2, 3\}$, and $\mathsf{graph}(r) = \{(Q, 2), (R, 3), (S, 1)\}$.

(e) $g : \mathbb{R} \to \mathbb{R}\,,\ \ \forall x \in \mathbb{R}\,, g(x) = x^3$.

HW2.2 Exercise 3 at the end of Chapter 2 of Lee & Varaiya. In addition, determine whether the following statement is true or false, and explain your reasoning.

(e) $\{g | g = \mathsf{graph}(f) \wedge f : \mathsf{X} \to \mathsf{Y}\} \subset \mathsf{P}(\mathsf{X} \times \mathsf{Y})$.

HW2.3 In this problem, we examine certain operations on signals, e.g., even and odd decomposition, time scaling (temporal expansion or contraction), shift (advance or delay), time reversal, pointwise multiplication, and convolution.

Let t be a fixed real number, and consider the signals $f : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$, where

$$\forall \tau \in \mathbb{R}\,, \quad f(\tau) = \begin{cases} 0 & \tau < 0 \\ e^{-\tau} & \tau \geq 0\,, \end{cases} \quad \text{and} \quad h(\tau) = \begin{cases} 1 & 0 \leq \tau \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Sketch $\mathsf{graph}(f)$ and $\mathsf{graph}(h)$.

(b) (Even and Odd Decomposition) Let $f_e : \mathbb{R} \to \mathbb{R}$ and $f_o : \mathbb{R} \to \mathbb{R}$ denote the even and odd components of f, respectively, where

$$\forall \tau \in \mathbb{R}\,, \qquad f_e(\tau) = \frac{f(\tau) + f(-\tau)}{2} \quad \text{and} \quad f_o(\tau) = \frac{f(\tau) - f(-\tau)}{2}\,.$$

Show that f_e and f_o are, indeed, even and odd, respectively. Express f in terms of f_e and f_o. Sketch $\mathsf{graph}(f_e)$ and $\mathsf{graph}(f_o)$. Can you reconstruct f from either f_e or f_o, but not both? Provide an example of a signal that cannot be reconstructed from only one of its even or odd constituents.

(c) (Temporal Expansion and Contraction) Define $\widehat{h}$ and $\widetilde{h}$ as follows: $\forall \tau \in \mathbb{R}, \widehat{h}(\tau) = h(2\,\tau)$ and $\widetilde{h}(\tau) = h(\tau/2)$. Sketch $\mathsf{graph}(\widehat{h})$ and $\mathsf{graph}(\widetilde{h})$.

(d) (Shift) Consider the signal $g : \mathbb{R} \to \mathbb{R}, \forall \tau \in \mathbb{R}, g(\tau) = f(\tau - t)$. Sketch a plot of g for $t = 1$ and $t = -1$. Observe that if $t > 0$, then $\mathsf{graph}(g)$ is obtained by moving $\mathsf{graph}(f)$ to the right (i.e., delaying f), and if $t < 0$, then $\mathsf{graph}(g)$ is obtained by moving $\mathsf{graph}(f)$ to the left (i.e., advancing f).

(e) (Time Reversal) Consider the signal $v : \mathbb{R} \to \mathbb{R}, \forall \tau \in \mathbb{R}, v(\tau) = f(-\tau)$. Sketch $\mathsf{graph}(v)$.

(f) Consider the signal $w : \mathbb{R} \to \mathbb{R}, \forall \tau \in \mathbb{R}, w(\tau) = f(t - \tau)$. For $t = 1$ and $t = -1$, provide a sketch of $\mathsf{graph}(w)$. Observe that if $t > 0$, then $\mathsf{graph}(w)$ is obtained by moving $\mathsf{graph}(v)$ of part (d) to the right, and if $t < 0$, then $\mathsf{graph}(w)$ is obtained by moving $\mathsf{graph}(v)$ to the left.

(g) (Pointwise Multiplication) Consider the signal $p : \mathbb{R} \to \mathbb{R}, \forall \tau \in \mathbb{R}, p(\tau) = h(\tau)\, w(\tau) = h(\tau)\, f(t - \tau)$. For each range of t, provide a sketch of $\mathsf{graph}(p)$: $t < 0, 0 < t < 1$, and $1 < t$.

(h) (Convolution) Consider the signal $q : \mathbb{R} \to \mathbb{R}$,

$$\forall t \in \mathbb{R}, \quad q(t) = \int_{-\infty}^{\infty} p(\tau)\, d\tau \; = \int_{-\infty}^{\infty} h(\tau)\, f(t - \tau)\, d\tau\,.$$

Provide a sketch of $\mathsf{graph}(q)$. The signal q results from the "convolution" of the signals h and f; this is written $q = h * f$ and, on a pointwise basis, $q(t) = (h * f)(t), \forall t \in \mathbb{R}$.

HW2.4 A *truth table* is an array depicting the truth values of assertions and their logical compositions. For example, a table depicting truth values for a conjunction and a disjunction is shown below:

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

By constructing an appropriate truth table, prove each of the following:

(a) *DeMorgan's Laws*

$$\begin{aligned} \neg(P \wedge Q) &\equiv \neg P \vee \neg Q \\ \neg(P \vee Q) &\equiv \neg P \wedge \neg Q\,, \end{aligned}$$

where the symbol "$\equiv$" denotes the logical equivalence of the compound assertions on its two sides.

(b) *Distributive Laws*

$$\begin{aligned} P \wedge (Q \vee R) &\equiv (P \wedge Q) \vee (P \wedge R) \\ P \vee (Q \wedge R) &\equiv (P \vee Q) \wedge (P \vee R) \end{aligned}$$

(c) The assertion "P, unless Q" is equivalent to the assertion "P or Q."

HW2.5 *Deductive Logic*

(a) Consider two assertions P and Q. We know that the following compound assertion is true:

$$(P \Rightarrow Q) \wedge (P \Rightarrow \neg Q)\,.$$

We do *not* know whether Q is true; it may or may not be true.

Choose the strongest statement below that most accurately reflects the truth or falsehood of P.

(I) P must be true.

(II) P can be true, but is not necessarily true.

(III) P must be false.

Explain your reasoning succinctly, but clearly and convincingly.

(b) Let $\mathrm{S}_{\mathrm{ZIZO}}$ be the symbolic representation of the assertion

″System S possesses the zero-input, zero-output (ZIZO) property″.

That is, ″System S is ZIZO.″

Also let S_{L} denote the assertion

″System S is linear.″

We know that the following assertion is true:

″System S is *not* linear, unless it is ZIZO.″

Suppose we know that the system S is ZIZO.

Can we conclude that S is linear? Explain your reasoning succinctly, but clearly and convincingly.

HW2.6 *Composition of Functions.* The following part are mutually independent.

(a) Consider the following two functions:

$$f : \mathbb{R} \to \mathbb{C} \ , \ \ \forall x \in \mathbb{R} \ , f(x) = \cos x + i \sin x \ ,$$

and

$$g : \mathbb{C} \to \mathbb{R} \ , \ \ \forall x \in \mathbb{C} \ , g(x) = \mathsf{Re}(x^3) \ ,$$

where $\mathsf{Re}(x)$ denotes the real part of x. Determine a simple expression for $(g \circ f)(x)$ in terms of x.

(b) Consider the functions $\mathbf{F}$ and $\mathbf{G}$ defined as follows:

$$\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2, \forall \boldsymbol{x} \in \mathbb{R}^2, \mathbf{F}(\boldsymbol{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \boldsymbol{x} \ ,$$

and

$$\mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^2, \forall \boldsymbol{z} \in \mathbb{R}^2, \mathbf{G}(\boldsymbol{z}) = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \boldsymbol{z} \ .$$

Determine $\mathbf{F} \circ \mathbf{G}$ and $\mathbf{G} \circ \mathbf{F}$. How are $\mathbf{F}$ and $\mathbf{G}$ related?

(c) Consider the function $h : \mathbb{R} \to \mathbb{R}, \forall x \in \mathbb{R}, h(x) = \cos(x)$. In this exercise, you will solve the transcendental equation $x = \cos(x)$ numerically by iteratively composing h with itself, i.e., you will find a numerical value for:

$$(h \circ h \circ h \cdots h)(x) = h(\cdots h(h(x)) \cdots) \ .$$

Using a computing device (e.g., a scientific calculator), enter an arbitrary initial (seed) value $x_0 \in \mathbb{R}$. With the calculator in "radians" mode, press the $\cos$ key once, to find the value of the first iterate $x_1 = \cos(x_0)$. Now, press the $\cos$ button repeatedly to produce values of the subsequent iterates $x_2 = \cos(x_1) = (\cos \circ \cos)(x_0) = \cos(\cos(x_0))$, $x_3 = \cos(x_2) = \cos(\cos(\cos(x_0)))$, $\ldots$, $x_{n+1} = \cos(x_n)$, $\ldots$. Alternatively, you may implement this with a simple LabView or MATLAB program.

What number $\overline{x}$ (up to three decimal digits of accuracy) do the calculations seem to converge to? Why is this a reasonable numerical solution to the transcendental equation $x = \cos(x)$?

This is an example of a dynamic system characterized by an *iterated map* of the form $x_{n+1} = h(x_n)$. The number $\overline{x}$ is called a *stable equilibrium point* of the iterated map. Later in the course, we will discover how the iterations converge to the value you found in your calculator experiment.